Solutions

Exam 1 Practice Problems

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1. Let $f(x) = \frac{1-\sqrt{x}}{1-x}$, $g(x) = \frac{\sin x}{x}$ and $h(x) = \frac{(2+x)^2-4}{x}$. Find the desired limits.

   (a) $\lim_{x\to 1} f(x)$.

   (b) $\lim_{x\to 0} g \cdot h(x)$.

   (c) $\lim_{x\to 2} \frac{f}{g}(x)$.

   (d) $\lim_{x\to 1} \frac{h}{f}(x)$.

   (e) $\lim_{x\to \pi} (f \cdot h - g \cdot f)(x)$.

(a) $f(x) = \frac{1-\sqrt{x}}{1-x} = \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})} = \frac{1}{1+\sqrt{x}}$ when $x \neq 1$.

So $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2}$.

(b) $h(x) = \frac{(2+x)^2-4}{x} = \frac{((2+x)-2)((2+x)+2)}{x} = \frac{x(x+4)}{x} = x+4$ when $x \neq 0$

Thus $\lim_{x\to 0} g \cdot h(x) = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} x+4 = 4$

(c) $\lim_{x\to 2} \frac{f}{g}(x) = \frac{f(2)}{g(2)} = \frac{+2(\sqrt{2}-1)}{\sin(2)}$.

(d) $\lim_{x\to 1} \frac{h}{f}(x) = \frac{\lim_{x\to 1} h(x)}{\lim_{x\to 1} f(x)} = \frac{5}{1/2} = 10$

(e) $\lim_{x\to \pi} (f\cdot h - g \cdot f)(x) = f(\pi)\cdot h(\pi) - g(\pi)\cdot f(\pi) = \frac{1-\sqrt{\pi}}{1-\pi}\left(\frac{(2+\pi)^2-4}{\pi} - 0\right)$

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2. Find explicitly the (best possible) continuous extension of f, g and h from problem 1.

$$F(x) = \begin{cases} \frac{1-\sqrt{x}}{1-x}, & x \neq 1 \\ 1/2, & x = 1 \end{cases}$$

$$G(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

$$H(x) = \begin{cases} \frac{(2+x)^2-4}{x}, & x \neq 0 \\ 4, & x = 0 \end{cases}$$

3. Use the intermediate value theorem.

(a) Find an interval on which $y = x^3 - x - 1$ has a zero.

(b) Show that the equation $\sqrt{2x+5} = 4 - x^2$ has a solution.

(c) Show that the equation $\cos x = x^2$ has a solution.

(d) Show that the graph of the equation $x^3 - 3x$ crosses the line $y = 1$.

(e*) Let f be a continuous function on the interval $[0,1]$. Suppose that $0 \le f(x) \le 1$ for every $x \in [0,1]$. Show that there must be a number $c \in [0,1]$ such that $f(c) = c$. (c is called a fixed point of f)

(a) $y(0) = -1$, $y(2) = 5$. So there is a root of y between 0 and 2.

(b) Consider $f(x) = \sqrt{2x+5} + x^2$. $f(0) = \sqrt{5} < 4$ and $f(2) = \sqrt{9} + 4 > 4$. So there is a solution between 0 and 2.

Solutions (c)-(e) on next page.

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4. Do the following problems involving limits at infinity.

(a) Find $\lim_{x\to\infty} x\frac{\sin 4/x}{2}$.

(b) Find all asymptotes of $y = \left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3}$.

(c) Find all asymptotes of $h(t) = \frac{t^3+7t^2-2}{t^2-t+1}$.

(d) Consider $g(x) = \left(\frac{x^2}{2} - \frac{1}{x}\right)$. Find the limits of $g(x)$ as $x \to 0^+$, $x \to 0-$, $x \to \sqrt[3]{2}$ and $x \to -1$.

(a) $\lim_{x\to\infty} \frac{x \cdot \sin(4/x)}{2} = \lim_{t\to 0} \frac{2\sin(t)}{t} = 2$ where $t = \frac{4}{x}$.

(b) $\lim_{x\to\infty} \left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3} = \left(\frac{1}{8}\right)^{1/3} = \frac{1}{2}$. Similarly $\lim_{x\to-\infty} \left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3} = \frac{1}{2}$. So hor. asymp. at $y = \frac{1}{2}$.

$8x^2 - 3 = 0$ when $x = \pm\sqrt{\frac{8}{3}}$. $\lim_{x\to\pm\sqrt{8/3}} y = \pm\infty$. So vert. asymp. at $x = \sqrt{8/3}$ and $x = -\sqrt{8/3}$

Solutions (c)-(d) on next page.

3.(c) Consider $g(x) = \cos x - x^2$. $g(0) = 1$ and $g(\pi/2) = -\frac{\pi^2}{4}$.

Thus there exists $c \in [0, \pi/2]$ such that $g(c) = 0$.

(d) Let $h(x) = x^3 - 3x$. Then $h(0) = 0$ and $h(2) = 2$. Thus there is a $c \in [0,2]$ such that $h(c) = 1$. So $x^3 - 3x$ passes through $y = 1$ etc.

(e) Let $g(x) = f(x) - x$. Then $g(0) = f(0) \in [0,1]$ and $g(1) = f(1) - 1 \leq 0$ since $f(1) \in [0,1]$. Thus there is some $c \in [0,1]$ such that $g(c) = 0$; i.e. $f(c) = c$.

4.(c) $h(t) = \frac{t^3 + 7t^2 - 2}{t^2 - t + 1}$. Degree of Numerator is 1 greater than degree of denominator.

So slant asymptote.

$$\begin{array}{r|l} & t + 8 + \text{remainder } 7t - 10. \\ \hline t^2 - t + 1 & t^3 + 7t^2 - 2 \\ & -(t^3 - t^2 + t) \\ \hline & 8t^2 - t - 2 \\ & -8(t^2 - t + 1) \\ \hline & 7t - 10 \end{array}$$

So slant asymptote is $y = t + 8$.

Also $t^2 - t + 1 = (t-2)(t+1) = 0$ when $t = 2$ or $t = -1$.

Vertical asymptotes at $t = 2$ and $t = -1$ by checking limits.

(d) $g(x) = \left(\frac{x^2}{2} - \frac{1}{x}\right) = \frac{x^3 - 2}{2x}$. $\lim_{x \to 0^+} g(x) = -\infty$. $\lim_{x \to 0^-} g(x) = \infty$

$\lim_{x \to \sqrt[3]{2}} g(x) = 0$ $\lim_{x \to -1} g(x) = \frac{3}{2}$.